JOURNAL OF DIFFERENTIAL EQUATIONS **127**, 439–453 (1996)
ARTICLE NO. 0077

Weak Solutions of Nonlinear Elliptic Equations with Prescribed Singular Set

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Received October 5, 1994; revised February 22, 1995

Given Ω any open and bounded subset of $\mathbb{R}^n$, $n \geqslant 4$, with smooth boundary and

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which are singular on Σ, when p is a real $p > m/(m-2)$, close to this value.

1. Introduction

Many authors have studied positive weak solutions of the equation

$$-\Delta u = u^p, \tag{1}$$

using either the geometrical or the analytic properties of this equation for special values of the exponent p. They looked for weak solutions with singular sets Λ which in general reduce to either a set of points or smooth submanifolds.

The case of the asymptotic behavior near an isolated singularity has been studied by P. Aviles (cf. [2]) when $p = (n/n-2)$, by B. Gidas and J. Spruck (cf. [7]) in the case where $p \in ((n/n-2), (n+2/n-2))$ and finally by L. A. Caffarelli, B. Gidas and J. Spruck, in their paper [4], for the case $p = (n+2)/(n-2)$, namely the equation

$$-\Delta u = u^{(n+2)/(n-2)}. \tag{2}$$

0022-0396/96 $18.00

They give some results about the asymptotic behavior of the singular solutions of (1).

Concerning the existence of singular solutions, F. Pacard, in his paper [12], has studied the problem

$$\begin{cases} -\Delta u = u^{(n/n-2)} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

He proved the existence of infinitely many positive weak solutions of (3) having prescribed singular set. He extended to other exponents this result in [13] for $(n/n-2) < p < (n+2)/(n-2)$ and he proved that there exist C_n, $(n/n-2) < C_n < (n+2)/(n-2)$ such that for all p satisfying $(n/n-2) < p < C_n$, for all $\Omega \subset \mathbb{R}^n$ and all $\{x_i\}_{i \in I}$ a countable set of isolated points there exist two distinct sequences of positive weak solutions of (1) whose singularities are exactly the points $\{x_i, i \in I\}$. Moreover one sequence of solutions converges to 0 in $L^p(\Omega)$ and the other converges to a regular solution of (1) in $L^p(\Omega)$. He remarked that his method fails for exponents larger than C_n and that he was not able to prove the existence of positive solutions of (1) whose singular set is given by ω for any subset $\omega \subset \Omega$ which is a result that holds for the equation (3). Recently, Chiun-Chuan Chen and Chang-Shou Lin, in their paper [5], proved the existence of infinite many positive weak solutions of the problem (1) with a given closed subset S of Ω as their singular set for the exponents p satisfying $(n/n-2) < p < (n+\sqrt{n-1})/(n-4+2\sqrt{n-1})$. They also constructed, as an application of their result to the conformal scalar curvature, a weak solution $u \in L^{(n+2)/(n-2)}(S^n)$ of the problem $L_0 u + u^{(n+2)/(n-2)} = 0$ for $n \geqslant 9$ such that S^n is the singular set of u, where L_0 is the conformal Laplacian with respect to the standard metric of S^n.

In his paper [16], R. Schoen studies the singular Yamabe problem which is a problem that concerns the existence of complete metrics g with constant scalar curvature on the complement of a closed subset Λ in a compact Riemannian manifold M and which are conformal to a metric smooth across the submanifold. This is equivalent to the problem of finding u positive, satisfying the equation

$$Lu + c(n)\, R(g)\, u^{(n+2)/(n-2)} = 0 \ \text{ in } \ M \backslash \Lambda, \text{g complete metric on } \ M \backslash \Lambda. \tag{4}$$

where $L = \Delta_{g_0} - c(n)\, R(g_0)$ is the conformal Laplacian with respect to the metric g_0, $c(n)$ is a constant depending on n, $g = u^{(4/n-2)} g_0$ and $R(g_0)$, the scalar curvature of g_0, is taken to be a constant. The geometric condition for the metric g to be complete is equivalent to the analytic one of requiring that u tends to $+\infty$ on average when x tends to Λ. R. Schoen constructs metrics with constant positive scalar curvature on S^n, conformal to the standard metric and with prescribed isolated singularities. He also constructs solutions with certain more general singular sets. R. Mazzeo, D. Pollack and K. Uhlenbeck (cf. [9]), have also studied the singular Yamabe problem where Λ is a finite set of points and gave some generalities on the set of solutions for this problem. They were also interested in the properties of the moduli space which is the set of all solutions u to the problem (4). D. Pollack (cf. [15]) was interested by the space of all solutions of the Yamabe problem which are singular on a prescribed set of points and he proved that any sequence of solutions has a subsequence convergent to a singular solution provided that certain natural invariants of these solutions are controlled. The control imposed on these invariants essentially guarantees that the singular points are "uniformly nonremovable".

Concerning the case of singular sets equal to smooth submanifolds of positive dimension, R. Schoen and S. T. Yau (cf. [17]) derived through ideas of conformal geometry the existence of weak solutions of the Eq. (2) having a singular set of Hausdorff dimension $m \leqslant (n-2)/2$. They explained how to build solutions with m not necessarily an integer. Recently, R. Mazzeo and N. Smale (cf. [10]) constructed many other examples when Λ is a k-dimensional submanifold of class $C^{3,\alpha}$ in the sphere near to any round S^k. Solutions with $R>0$ are also constructed in the tubular neighborhood of any submanifold of dimension k, $1 \leqslant k \leqslant (n-2)/2$, in any manifold of positive scalar curvature. This settled, at least locally, the question whether there could exist solutions to the problem (2) with $R>0$ singular along submanifolds other than round subspheres and it also pointed out the extreme nonuniqueness inherent in the problem, since the solution spaces are infinite dimensional for fixed Λ. All these solutions are obtained perturbation-theoretically starting with exact or approximate solutions. F. Pacard, in his paper [14], has studied the Eq. (2) and he proved the existence of a weak solution of the Eq. (2) with a singular set Λ equal to a finite union of $((n-2)/2)$-dimensional compact submanifolds without boundaries of $\mathbb{R}^n$. The resolution of this equation allowed him to resolve the problem $L_0 U = U^{(n+2)/(n-2)}$ on $S^n \backslash \Lambda'$ where $L_0 U = \Delta_{g_0} U - n(n-2)/4\ U$, Δ_{g_0} denotes the Laplacian with respect to g_0 the standard metric on S^n and Λ' is a finite union of compact $((n-2)/2)$-dimensional submanifolds of S^n without boundaries and such that Λ is its stereographic projection. This result ensures the existence of a complete metric on $S^n \backslash \Lambda'$ with constant positive scalar curvature conformally equivalent to the standard metric on $S^n \backslash \Lambda'$. In another paper [11], the same author studies the Eq. (3) and proves the existence of a weak solution with prescribed singular set in the following sense: for any closed set $S \subset \Omega$ there exists a positive weak solution of (3) whose singular set is given by S.

Our aim in this paper is to prove the existence of positive weak solutions to the equation (1), with $p > (m/m-2)$ and very close to this value such that its singular set is a given $(n-m)$-dimensional compact submanifold of Ω without boundary. The method used to this end is a mixture of arguments used by both R. Mazzeo and N. Smale in their paper [10] and by F. Pacard in his paper [14].

2. Main Result

Let Ω be any open and bounded subset of $\mathbb{R}^n$ with smooth boundary, $n \geqslant 4$. Assume that $2 < m < n$ and let the exponent p be chosen such that $p > (m/m-2)$. We fix m and n and we consider Σ any compact submanifold of Ω without boundary of dimension $(n-m)$. We have the following theorem:

THEOREM 1. *Given Ω an open subset of $\mathbb{R}^n$ with a smooth boundary, Σ a compact submanifold of Ω, $n \geqslant 4$, $2 < m < n$ and $p > m/(m-2)$; the problem*:

$$(P_\Sigma) \begin{cases} -\Delta u = u^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

has at least one weak solution whose singular set is given by Σ.

We give now an outline of the proof: We first look for radial solutions of the problem $-\Delta u = u^p$ in $\mathbb{R}^m$. Using this first step we construct an approximate solution. We then conclude studying the nonlinear problem and using the Schauder fixed point theorem to get a solution to (5) as a perturbation of the approximate solution.

In our problem, the linearized operator near an approximate solution does not have any kernel in the space of functions we consider to solve this problem, thanks to this property, our proof of the existence of a solution is considerably simpler than the proof in [10].

3. Proof of the Result

3.1. *Study of Radial Solutions*

We look for radial solutions of the problem $-\Delta u = u^p$ in $\mathbb{R}^m$. We set $t = -\log r$ where $r = |x|$ and we set $v(t) = r^{2/(p-1)}\, u(x)$. If u is a radial solution of the problem $-\Delta u = u^p$ in $\mathbb{R}^m$, then, after a computation, we see that v satisfies

$$v'' + \left(\frac{4}{p-1} + (2-m)\right) v' - \frac{2}{p-1}\left(m - \frac{2p}{p-1}\right) v + v^p = 0, \tag{6}$$

where v' and v'' are respectively the first and the second t-derivative of v.

By a classical study of Eq. (6), cf. [4], we conclude that there exists a singular radial solution u_0 of the problem $-\Delta u = u^p$ in $\mathbb{R}^m$ such that

$$u_0(x) = \begin{cases} (C_p + o(1))\, |x|^{-2/(p-1)} & \text{near } 0 \\ (1 + o(1))\, |x|^{2-m} & \text{near } +\infty. \end{cases} \tag{7}$$

Where C_p is given by $(2/(p-1)(m-2p/(p-1))^{1/(p-1)}$. We notice that C_p tends to zero when p tends to $m/(m-2)$. In addition, there exists some constant c_p which converges to 0 as p tends to $m/(m-2)$ such that

$$u_0(x) \leqslant c_p\, |x|^{-2/(p-1)}. \tag{8}$$

3.2. *Construction of the Approximate Solution*

Let $\sigma > 0$ be given. We consider a σ-tubular neighborhood of Σ in $\mathbb{R}^n$, that we denote by $N(\sigma, \Sigma)$ and we let $\bar{N}(\sigma, \Sigma)$ be its closure. For $x \in \bar{N}(\sigma, \Sigma)$, let $r = \text{dist}(x, \Sigma)$. Fixing σ small enough, we can locally identify $\bar{N}(\sigma, \Sigma)$ with a bundle of radius σ; that is, for every $y_0 \in \Sigma$, there exist a neighborhood $\mathscr{U}$ of y_0 in Σ, such that $\bar{N}(\sigma, \Sigma)$ is locally equivalent to the trivial bundle $B^m(\sigma) \times \mathscr{U}$, where $B^m(\sigma)$ is the ball of radius σ centered at 0 in $\mathbb{R}^m$. Thus we can have the following local coordinate system for $\bar{N}(\sigma, \Sigma)$ near y_0 (cf. R. Mazzeo and N. Smale [10]). We use polar coordinates (r, θ) on $B^m(\sigma)$ so that (r, θ, y) are coordinates on $\bar{N}(\sigma, \Sigma)$. We consider $\varepsilon > 0$. Using the function u_0 defined in the previous subsection, we build u_ε such that

$$u_\varepsilon(x) = \varepsilon^{-2/(p-1)}\, u_0\left(\frac{x}{\varepsilon}\right) \qquad \text{for} \quad x \in \mathbb{R}^m. \tag{9}$$

We remark that we still have $-\Delta_m u_\varepsilon = u_\varepsilon^p$ thanks to the fact that the equation is invariant under dilation.

Let χ be a smooth function from $\mathbb{R}^m$ to $\mathbb{R}^+$ such that:

$$\begin{cases} \chi \equiv 1 & \text{on } B^m(\frac{1}{2}) \\ \chi \equiv 0 & \text{on } \mathbb{R}^m \backslash B^m(1) \\ 0 \leqslant \chi \leqslant 1 & \text{in } \mathbb{R}^m. \end{cases} \tag{10}$$

For all $\tau > 0$, we define $\chi_\tau(x) = \chi(\frac{x}{\tau})$ and we set, for $(x, y) = (r, \theta, y) \in \bar{N}(\sigma, \Sigma)$,

$$\tilde{u}_\varepsilon(x, y) \equiv \tilde{u}_\varepsilon(x) \equiv \varepsilon^{-2/(p-1)}\, u_0\left(\frac{x}{\varepsilon}\right) \chi_\sigma(x). \tag{11}$$

This construction allows us to define locally the function $\tilde{u}_\varepsilon$ in the neighborhood of all the points of $\bar{N}(\sigma, \Sigma)$.

We define Hölder spaces $\mathscr{C}^{k,\alpha,\nu}(\Omega)$ by:

$$\mathscr{C}^{k,\alpha,\nu}(\Omega)=\left\{u\in\mathscr{C}^{k,\alpha}_{loc}\mid \|u\|_{k,\alpha,\nu}<\infty\right\}. \tag{12}$$

Where $\|\cdot\|_{k,\alpha,\nu}$ is the norm:

$$\|u\|_{k,\alpha,\nu}=\sup_{0<s\leqslant|\Omega|} s^{-\nu}\,|u|_{k,\alpha;[s,2s]}, \tag{13}$$

where the norm on the right is the $\mathscr{C}^{k,\alpha}$ norm restricted to the set $\{(r,\theta,y: s\leqslant r\leqslant 2s)\}$ with respect to the underlying metric. i.e.:

$$\begin{aligned}|u|_{k,\alpha,[s,2s]}=&\sup_{dist(x,\Sigma)\in[s,2s]}\left(\sum_{j=0}^{k} r^j\,|\nabla^j u(x)|\right)\\ &+s^{k+\alpha}\sup_{dist(x_i,\Sigma)\in[s,2s]}\left(\frac{|\nabla^k u(x_i)-\nabla^k u(x_j)|}{|x_i-x_j|^\alpha}\right).\end{aligned} \tag{14}$$

Thus, when we work in $\mathscr{C}^{0,\alpha}$, the norm we consider is

$$|u|_{0,\alpha,[s,2s]}=\sup_{dist(x,\Sigma)\in[s,2s]}|u(x)|+s^\alpha\sup_{dist(x_i,\Sigma)\in[s,2s]}\frac{|u(x_1)-u(x_2)|}{|x_1-x_2|^\alpha}. \tag{15}$$

We have the following proposition

PROPOSITION 1 [10], [6]. *Let $(x,y)\in B^m(\sigma)\times\mathscr{U}$ be the induced coordinates for $\bar{N}(\sigma,\Sigma)$ near y_0. Then, for $u\in\mathscr{C}^{2,\alpha}(\bar{N}(\sigma,\Sigma))$, we have*

$$\Delta u=\Delta_{B^m}u+\Delta_\Sigma u+e_1\cdot\nabla^2 u+e_2\cdot\nabla u \tag{16}$$

on $B^m(\sigma)\times\mathscr{U}$, where ∇^2 denotes the Hessian, ∇ denotes the gradient on $\bar{N}(\sigma,\Sigma)$ i.e. we have:

$$\begin{cases} e_1\cdot\nabla^2 u\equiv\Sigma_{i,j}e_1^{ij}\dfrac{\partial^2 u}{\partial x_i\partial x_j}\\ e_2\cdot\nabla u\equiv\Sigma_i e_2^i\dfrac{\partial u}{\partial x_i}\end{cases}$$

and e_1 and e_2 are $\mathscr{C}^{0,\alpha}$ sections of $(Sym^2\,N(\sigma,\Sigma))^*$ and $TN(\sigma,\Sigma)$ such that

$$\|e_1\|_{0,\alpha,1}+\|e_2\|_{0,\alpha,0}\leqslant C_0, \tag{17}$$

for some constant C_0 independent of x, y, or σ and for any α satisfying $0<\alpha<1$.

For more details we refer to the paper of R. Mazzeo and N. Smale [10] or to the paper of D. R. Finn and R. C. McOwen [6] who gave an explicit proof of this local expression of the Laplacian. In what follows, we will denote $\Delta_x = \Delta_{B^m}$ and $\Delta_y = \Delta_\Sigma$. We prove that $\tilde{u}_\varepsilon$ is an approximate solution of the problem (P_Σ) in the following sense

PROPOSITION 2. *Assume that* α *and* v *are chosen to satisfy*

$$0 < \alpha < 1 \quad \textit{and} \quad -\frac{2}{p-1} < v < \frac{p-3}{p-1}.$$

Then we have

$$\lim_{\varepsilon \to 0} \|(\Delta \tilde{u}_\varepsilon + \tilde{u}_\varepsilon^p)\|_{0,\,\alpha,\,v-2} = 0.$$

Proof. Using the local expression of the Laplacian (16), we can write

$$\begin{aligned}\Delta \tilde{u}_\varepsilon + \tilde{u}_\varepsilon^p &= (u_\varepsilon \chi_\sigma)^p + \Delta_x (u_\varepsilon \chi_\sigma) + e_1 \cdot \nabla^2 \tilde{u}_\varepsilon + e_2 \cdot \nabla \tilde{u}_\varepsilon \\ &= u_\varepsilon^p (\chi_\sigma^p - \chi_\sigma) + 2\nabla(\chi_\sigma)\, \nabla(u_\varepsilon) + u_\varepsilon \Delta \chi_\sigma + e_1 \cdot \nabla^2 \tilde{u}_\varepsilon + e_2 \cdot \nabla \tilde{u}_\varepsilon.\end{aligned}$$

On the other hand, we know that

$$Supp\{(\chi_\sigma^p - \chi_\sigma)\, u_\varepsilon^p\} \subset B_\sigma \backslash B_{\sigma/2},$$
$$Supp\{\nabla \chi_\sigma \nabla u_\varepsilon\} \subset B_\sigma \backslash B_{\sigma/2}, \quad \text{and} \quad Supp\{u_\varepsilon \Delta \chi_\sigma\} \subset B_\sigma \backslash B_{\sigma/2}.$$

We also know that the problem is invariant under dilation. Therefore, if ε is very small, we have $u_\varepsilon \leqslant \varepsilon^{m-2p/(p-1)}\, |x|^{-m+2}$ in $B_\sigma \backslash B_{\sigma/2}$ (cf. (9)). Then we have

$$\begin{aligned} r^{2-v}\, & |u_\varepsilon^p (\chi_\sigma^p - \chi_\sigma) + 2\nabla(\chi_\sigma)\, \nabla(u_\varepsilon) + u_\varepsilon \Delta \chi_\sigma| \\ & \leqslant r^{2-v} (|\chi_\sigma^p - \chi_\sigma|\, \varepsilon^{p(m-2p/(p-1))}\, |x|^{p(2-m)} \\ & \quad + 2\, |\nabla \chi_\sigma|\, \varepsilon^{m-2p/(p-1)} (m-2)\, |x|^{1-m} + |\Delta \chi_\sigma|\, \varepsilon^{m-2p/(p-1)}\, |x|^{2-m}) \\ & \leqslant r^{2-v} \left(\varepsilon^{p(m-2p/(p-1))} \left(\frac{\sigma}{2}\right)^{p(2-m)} + 2 \varepsilon^{m-2p/(p-1)} (m-2) \left(\frac{\sigma}{2}\right)^{1-m} \right. \\ & \quad \left. + \varepsilon^{m-2p/(p-1)} \left(\frac{\sigma}{2}\right)^{2-m} \right). \end{aligned}$$

We also have

$$\begin{aligned} r^{2-v+\alpha}\, & |x_i - x_j|^{-\alpha}\, |(u_\varepsilon^p (\chi_\sigma^p - \chi_\sigma) + 2\nabla(\chi_\sigma)\, \nabla(u_\varepsilon) + u_\varepsilon \Delta \chi_\sigma)(x_i) \\ & - (u_\varepsilon^p (\chi_\sigma^p - \chi_\sigma) + 2\nabla(\chi_\sigma) \nabla(u_\varepsilon) + u_\varepsilon \Delta \chi_\sigma)(x_j)| \\ \leqslant r^{2-v+\alpha} c r^{-\alpha} & \left(\varepsilon^{p(m-2p/(p-1))} \left(\frac{\sigma}{2}\right)^{p(2-m)} + 2 \varepsilon^{m-2p/(p-1)} (m-2) \left(\frac{\sigma}{2}\right)^{1-m} \right. \\ & \left. + \varepsilon^{m-2p/(p-1)} \left(\frac{\sigma}{2}\right)^{2-m} \right). \end{aligned}$$

So, we deduce that

$$\|r^2(u_\varepsilon^p(\chi_\sigma^p - \chi_\sigma) + 2\nabla(\chi_\sigma)\,\nabla(u_\varepsilon) + u_\varepsilon \Delta\chi_\sigma)\|_{0,\alpha,v}$$
$$\leqslant Cs^{2-v}\left(\varepsilon^{p(m-2p/(p-1))}\left(\frac{\sigma}{2}\right)^{p(2-m)} + 2\varepsilon^{m-2p/(p-1)}(m-2)\left(\frac{\sigma}{2}\right)^{1-m} + \varepsilon^{m-2p/(p-1)}\left(\frac{\sigma}{2}\right)^{2-m}\right).$$

We are interested by the determination of an upper bound for $|e_1 \cdot \nabla^2 \tilde{u}_\varepsilon|$ and for $|e_2 \cdot \nabla \tilde{u}_\varepsilon|$. For this, we need to distinguish the case where $|x| < \varepsilon$ and then the case where $\sigma > |x| > \varepsilon$. (Recall that $Supp\{\tilde{u}_\varepsilon\} \subset B_m(\sigma)$).

First Case: $|x| < \varepsilon$. From (7), we know that

$$|\nabla u_0| \leqslant \tilde{c}_p\,|x|^{-(p+1)/(p-1)}.$$

So using (11), we can estimate

$$|e_2 \cdot \nabla \tilde{u}_\varepsilon| \leqslant C\tilde{c}_p\,|x|^{-(p+1)/(p-1)}.$$

Therefore, we conclude that, for all $|x| < \varepsilon$ we have

$$|r^{2-v} e_2 \cdot \nabla \tilde{u}_\varepsilon| \leqslant \tilde{c}_p r^{(p-3)/(p-1)-v}.$$

As we have assumed that $v < (p-3)/(p-1)$, we have

$$|r^{2-v} e_2 \cdot \nabla \tilde{u}_\varepsilon| \leqslant \tilde{c}_p \varepsilon^{(p-3)/(p-1)-v},$$

which tends to 0 when ε tends to 0. We also have

$$|e_2 \cdot \nabla \tilde{u}_\varepsilon(x_i) - e_2 \cdot \nabla \tilde{u}_\varepsilon(x_j)|$$
$$= |(e_2(x_i) \cdot \nabla \tilde{u}_\varepsilon(x_i) - e_2(x_i) \cdot \nabla \tilde{u}_\varepsilon(x_j)) + (e_2(x_i) \cdot \nabla \tilde{u}_\varepsilon(x_j) - e_2(x_j) \cdot \nabla \tilde{u}_\varepsilon(x_j))|$$
$$\leqslant C\,|\nabla^2 \tilde{u}_\varepsilon|\;|x_i - x_j| + C \sup_x |\nabla \tilde{u}_\varepsilon|.$$

Then we deduce

$$r^{2-v+\alpha}\frac{|e_2 \cdot \nabla \tilde{u}_\varepsilon(x_i) - e_2 \cdot \nabla \tilde{u}_\varepsilon(x_j)|}{|x_i - x_j|^\alpha}$$
$$\leqslant r^{2-v+\alpha}\,\|e_2\|\left(|x_i - x_j|^{1-\alpha}\,c_p r^{-2p/(p-1)} + \frac{r^{-(p+1)/(p-1)}}{|x_i - x_j|^\alpha}\right).$$

So, we have

$$r^{2-\nu+\alpha}\frac{|e_2\cdot\nabla\tilde{u}_\varepsilon(x_i)-e_2\cdot\nabla\tilde{u}_\varepsilon(x_j)|}{|x_i-x_j|^\alpha}\leqslant\tilde{c}_p r^{(p-3)/(p-1)-\nu},$$

with $\tilde{c}_p$ tending to 0 when p tends to $m/(m-2)$.

Using a similar argument, we can easily derive the estimate

$$\begin{aligned}|r^{2-\nu}e_1\cdot\nabla^2\tilde{u}_\varepsilon|\leqslant r^{2-\nu}rC\tilde{c}_p\,|x|^{-2p/(p-1)}&<\breve{C}_p r^{(p-3)/(p-1)-\nu}\\&\leqslant\breve{C}_p\varepsilon^{(p-3)/(p-1)-\nu},\end{aligned}$$

since we also assume that $|x|<\varepsilon$. We notice that $\breve{C}_p$ tends to 0 when p tends to $m/(m-2)$.

Second Case: $\varepsilon<|x|<\sigma$. From (7) we know that

$$|\nabla u_0(x)|\leqslant|x|^{1-m}$$

and that

$$|\nabla^2 u_0(x)|\leqslant|x|^{-m}.$$

Then we deduce the estimates

$$r^{2-\nu}\,|e_2\cdot\nabla\tilde{u}_\varepsilon(x)|\leqslant Cr^{2-\nu}\varepsilon^{-2/(p-1)-1+(m-1)}\,|x|^{1-m}\leqslant Cr^{3-m-\nu}\varepsilon^{m-2p/(p-1)}$$

and

$$r^{2-\nu}\,|e_1\cdot\nabla^2\tilde{u}_\varepsilon(x)|\leqslant Cr^{2-\nu}r\varepsilon^{-2/(p-1)-2+m}\,|x|^{-m}\leqslant Cr^{3-m-\nu}\varepsilon^{m-2p/(p-1)}.$$

We assumed that $\varepsilon<r<\sigma$ and $-2/(p-1)<\nu<(p-3)/(p-1)$, we now discuss different cases depending on the sign of $3-m-\nu$:

If we assume that $3-m-\nu\geqslant 0$, then

$$r^{2-\nu}(|e_2\cdot\nabla\tilde{u}_\varepsilon(x)|+|e_1\cdot\nabla^2\tilde{u}_\varepsilon(x)|)\leqslant C\sigma^{3-m-\nu}\varepsilon^{m-2p/(p-1)}.$$

If we assume that $3-m-\nu\leqslant 0$, we obtain

$$r^{2-\nu}(|e_2\cdot\nabla\tilde{u}_\varepsilon(x)|+|e_1\cdot\nabla^2\tilde{u}_\varepsilon(x)|)\leqslant C\varepsilon^{3-m-\nu}\varepsilon^{m-2p/(p-1)}\leqslant C\varepsilon^{(p-3)/(p-1)-\nu}.$$

So, $\tilde{u}_\varepsilon$ is an approximate solution in the sense described above. ∎

3.3. *The Nonlinear Problem*

We look for solutions of the problem which are obtained as a perturbation of $\tilde{u}_\varepsilon$ by a lower order term. Hence we linearize $u\mapsto\Delta u+u^p$ about

$\tilde{u}_\varepsilon$ and make a Taylor expansion to get for all $u \in \mathscr{C}^{2,\alpha,\nu}(\Omega)$ the nonlinear equation

$$-(\Delta u + p\tilde{u}_\varepsilon^{p-1} u) = (\Delta \tilde{u}_\varepsilon + \tilde{u}_\varepsilon^p) + \int_0^1 p(p-1)(1-t)\, |\tilde{u}_\varepsilon + tu|^{p-2}\, u^2\, dt. \qquad (18)$$

Remark 1. In the case studied by R. Mazzeo and N. Smale (cf. [10]), the linear operator $T = (-\Delta - p\tilde{u}_\varepsilon^{p-1})$ may have a kernel in the weighted Hölder space $\mathscr{C}^{2,\alpha,\nu}$ which does not allow one to use the maximum principle; but in our case, choosing p sufficiently close to $m/(m-2)$, we can show that the operator T has no kernel.

PROPOSITION 3. *Let ν be given such that $2-m<\nu<0$ and consider the operator $\mathbb{L}$ defined from the space $\mathscr{C}^{0,\alpha,\nu-2}(\Omega)$ into the space $\mathscr{C}^{2,\alpha,\nu}(\Omega)$ by $\mathbb{L}(f) = v$ where v solves*

$$\begin{cases} -\Delta v = f & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Then the operator $\mathbb{L}$ is bounded.

Proof. We first consider the problem

$$(\Pi)\colon \begin{cases} -\Delta v = f & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \qquad (19)$$

By Green's formula (see for example [8]) we get, from the maximum principle

$$\begin{aligned} |v(x)| &\leqslant \frac{1}{n(n-2)\,\omega_n} \int_\Omega |x-y|^{2-n}\, |\Delta v(y)|\, dy \\ &\leqslant \frac{1}{n(n-2)\,\omega_n} \int_\Omega |x-y|^{2-n}\, |f(y)|\, dy. \end{aligned}$$

Considering the case where $x \in \{y \in \Omega / dist(y, \Sigma) > \delta\}$, the above estimate allows us to conclude that

$$|v(x)| \leqslant \frac{1}{n(n-2)\,\omega_n}\, \frac{1}{\delta^{n-2}}\, \|f\|_{L^1(\Omega)}.$$

We notice that the norm in the right hand side is well defined since f belongs to $L^1(\Omega)$. Namely, using the fact that $f \in \mathscr{C}^{0,\alpha,\nu-2}$, we get the bound

$$|f(x)| \leqslant C\, |x|^{\nu-2}$$

and as we have chosen ν satisfying $2-m<\nu$, we conclude easily that $f \in L^1(\Omega)$. We now consider the tubular neighborhood of Σ defined by $\{y \in \Omega / dist(y, \Sigma) \leqslant \delta\}$. Let $w_0 = r^\nu$, we have $\omega_0 \in \mathscr{C}^{2,\alpha,\nu}$ and $\|\omega_0\|_{2,\alpha,\nu} \leqslant c$ where c is a positive constant. Using the local expression of the Laplacian (cf. (16)), we may write

$$\Delta_n w_0 = \Delta_x w_0 + \Delta_y w_0 + e_1 \cdot \nabla^2 w_0 + e_2 \cdot \nabla w_0.$$

We find, after a straightforward computation,

$$\Delta_n w_0 = \nu(\nu+m-2) r^{\nu-2} + e_1 \cdot \nabla^2 r^\nu + e_2 \cdot \nabla r^\nu.$$

Since, for $r<\delta$ small enough, $e_1 \cdot \nabla^2 r^\nu + e_2 \cdot \nabla r^\nu$ is negligible compared to $r^{\nu-2}$, we have

$$-\Delta_n r^\nu = -C_\nu r^{\nu-2} + o(r^{\nu-2}) \geqslant (-C_\nu/2)\, r^{\nu-2},$$

where the constant C_ν is given by $C_\nu = \nu(\nu+m-2)$. We notice that $C_\nu<0$ thanks to the assumption $2-m<\nu<0$. So,

$$-\Delta((-C_\nu/2)^{-1} r^\nu) \geqslant r^{\nu-2} \geqslant Cf.$$

We conclude that $\tilde{w}_0 = (-C_\nu/2)^{-1}\, r^\nu$ is a supersolution of (19) and we have $v \leqslant c_\nu w_0$. We can similarly prove that $-\tilde{w}_0$ is a subsolution. So, we have $|v| \leqslant c_\nu\, |w_0|$. Using rescalled Schauder estimates (cf. [8]), we conclude

$$\|\mathbb{L}(f)\|_{2,\alpha,\nu} \leqslant C(\|\mathbb{L}(f)\|_{0,\alpha,\nu} + \|f\|_{0,\alpha,\nu-2}) < C'\|f\|_{0,\alpha,\nu-2}.$$

where C and C' are two real constants. Then the operator $\mathbb{L}$ is bounded. ∎

Corollary 1. *Let ν be fixed such that $2-m<\nu<\inf(0, 3-m)$ and let $\mathbb{K}$ be defined from the space $\mathscr{C}^{2,\alpha,\nu}(\Omega)$ into the space $\mathscr{C}^{2,\alpha,\nu}(\Omega)$ by $\mathbb{K}(w) = \mathbb{L}(p\tilde{u}_\varepsilon^{p-1} w)$. Then, the norm of this operator tends to 0 as p tends to $m/(m-2)$.*

Proof. In order to prove the result we decompose $\mathbb{K}$ in the following way:

$$\begin{array}{llll} & \quad\mathbb{M} & \quad\mathbb{L} & \\ \mathbb{K}: \mathscr{C}^{2,\alpha,\nu}(\Omega) & \to \mathscr{C}^{0,\alpha,\nu-2}(\Omega) & \to \mathscr{C}^{2,\alpha,\nu}(\Omega) & \\ \quad w & \mapsto p\tilde{u}_\varepsilon^{p-1} w & \mapsto \mathbb{L}(p\tilde{u}_\varepsilon^{p-1} w). & \end{array}$$

On one hand, we notice that both operators $\mathbb{M}$ and $\mathbb{L}$ are bounded into the corresponding spaces, so $\mathbb{K}$ is bounded too. On the other hand, $\mathbb{M}$ has a small norm tending to zero as p tends to $m/(m-2)$ thanks to its linearity and to the estimate (8) so that the norm of $\mathbb{L}$ composed with $\mathbb{M}$ tends to zero as p tends to $m/(m-2)$. ▮

Proof of Remark 1. If the norm of the operator $\mathbb{K}$ is small, then we conclude that the operator $T=(-\Delta-p\tilde{u}_\varepsilon^{p-1})$ is invertible so it has not a kernel i.e. there exist no $w\not\equiv 0$ such that $Tw=0$. ▮

Let us return to our problem. We now fix ν satisfying

$$2-m<\nu<3-m.$$

We are going to prove that the result of Theorem 1 holds for p close enough to the value $m/(m-2)$.

4. Proof of Theorem 1

We want to solve the problem

$$\begin{cases} -\Delta u=\Delta\tilde{u}_\varepsilon+|\tilde{u}_\varepsilon+u|^p & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{20}$$

which is equivalent to the problem

$$\begin{cases} -\Delta u=(\Delta\tilde{u}_\varepsilon+\tilde{u}_\varepsilon^p)+|\tilde{u}_\varepsilon+u|^p-\tilde{u}_\varepsilon^p & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{21}$$

For this, we shall prove that the operator $\mathbb{P}$ defined from $\mathscr{C}^{2,\alpha,\nu}(\Omega)$ into $\mathscr{C}^{2,\alpha,\nu}(\Omega)$ by

$$\mathbb{P}(u)=\mathbb{L}((\Delta\tilde{u}_\varepsilon+\tilde{u}_\varepsilon^p)+|\tilde{u}_\varepsilon+u|^p-\tilde{u}_\varepsilon^p),$$

is a contraction. We will reduce our study to B_θ the ball of radius θ of $\mathscr{C}^{2,\alpha,\nu}(\Omega)$. We know that $(\Delta\tilde{u}_\varepsilon+\tilde{u}_\varepsilon^p)$ tends to zero in $\mathscr{C}^{0,\alpha,\nu-2}$ as ε tends to zero (cf. Proposition 2), so $\mathbb{L}(\Delta\tilde{u}_\varepsilon+\tilde{u}_\varepsilon^p)$ tends to zero in $\mathscr{C}^{2,\alpha,\nu}$ as ε tends to zero. We have to study $\mathbb{L}(|\tilde{u}_\varepsilon+u|^p-\tilde{u}_\varepsilon^p)$ for $u\in B_\theta$. Let us consider the operator $\mathbb{N}$ defined from $\mathscr{C}^{2,\alpha,\nu}$ into $\mathscr{C}^{0,\alpha,\nu-2}$ by $\mathbb{N}(u)=|\tilde{u}_\varepsilon+u|^p-\tilde{u}_\varepsilon^p$. We will use the following lemma

Lemma 1. *Given x and y two real numbers, $x>0$ and for all small $\eta\in\mathbb{R}$ there exists a positive constant C_η such that*

$$\big||x+y|^p-x^p\big|\leqslant(1+\eta)\,px^{p-1}\,|y|+C_\eta\,|y|^p.$$

Thanks to this result, we obtain

$$||\tilde{u}_\varepsilon+u|^p-\tilde{u}_\varepsilon^p| \leqslant (1+\eta)\, p\tilde{u}_\varepsilon^{p-1}\,|u|+C_\eta\,|u|^p.$$

So we deduce that

$$\begin{aligned}\|\mathbb{L}\mathbb{N}(u)\|_{2,\alpha,\nu} &\leqslant (1+\eta)\ \|\mathbb{K}(u)\|_{2,\alpha,\nu}+C_\eta\ |||\mathbb{L}|||\ \|u\|^p_{2,\alpha,\nu}\\ &\leqslant (1+\eta)\ |||\mathbb{K}|||\ \|u\|_{2,\alpha,\nu}+C_\eta\ |||\mathbb{L}|||\ \|u\|^p_{2,\alpha,\nu},\end{aligned}$$

where $|||\cdot|||$ is the norm of the considered operator in the corresponding spaces. As we have chosen $u\in B_\theta$, we claim that

$$\|\mathbb{L}[\,|\tilde{u}_\varepsilon+u|^p-\tilde{u}_\varepsilon^p]\|_{2,\alpha,\nu} \leqslant ((1+\eta)\ |||\mathbb{K}|||+C_\eta\theta^{p-1})\ \|u\|_{2,\alpha,\nu}.$$

Knowing that ν is fixed, that, thanks to the Corollary 1, for a suitable value of p close to $m/(m-2)$ we can claim that $|||\mathbb{K}|||<1$ and that for a suitably small value of η we can have $(1+\eta)\,|||\mathbb{K}|||<1$, we deduce that if we choose θ very small, we can have $(1+\eta)\,|||\mathbb{K}|||+C_\eta\theta^{p-1}\leqslant 1$ so that

$$\|\mathbb{P}(u)\|_{2,\alpha,\nu}\leqslant\theta \qquad \text{for all} \quad u\in B_\theta.$$

We conclude that the operator $\mathbb{P}$ maps B_θ into itself. We have to prove that it is a contraction from B_θ to itself. Let v_1 and v_2 be in B_θ. We have

$$\|\mathbb{P}(v_1)-\mathbb{P}(v_2)\|_{2,\alpha,\nu}=\|\mathbb{L}[\,|\tilde{u}_\varepsilon+v_1|^p-|\tilde{u}_\varepsilon+v_2|^p]\|_{2,\alpha,\nu}.$$

So using the Schauder estimate (cf. [8]), we have on one hand

$$\|\mathbb{P}(v_1)-\mathbb{P}(v_2)\|_{2,\alpha,\nu}\leqslant C(\|\mathbb{P}(v_1)-\mathbb{P}(v_2)\|_{0,\alpha,\nu}+\|\,|\tilde{u}_\varepsilon+v_1|^p-|\tilde{u}_\varepsilon+v_2|^p\,\|_{0,\alpha,\nu}).$$

But, on the other hand, thanks to the Lemma 1, we have

$$\begin{aligned}&\big|\,|\tilde{u}_\varepsilon+v_1|^p-|\tilde{u}_\varepsilon+v_2|^p\big|\\ &\quad\leqslant (1+\eta)p\ |\tilde{u}_\varepsilon+v_1|^{p-1}\ |v_1-v_2|+C_\eta\ |v_1-v_2|^p\\ &\quad\leqslant |v_1-v_2|\ (1+\eta)p\ [(1+\eta)\ |\tilde{u}_\varepsilon|^{p-1}+C_\eta\ |v_1|^{p-1}]+C_\eta\ |v_1-v_2|^p.\end{aligned}$$

Then if we use this estimate for $\mathbb{P}(v_1)-\mathbb{P}(v_2)$ we obtain

$$\begin{aligned}\|\mathbb{P}(v_1)-\mathbb{P}(v_2)\|_{2,\alpha,\nu}\leqslant C(|||\mathbb{L}|||\ \|v_1-v_2\|_{0,\alpha,\nu}\,&[(1+\varepsilon)^2p\ \|\tilde{u}_\varepsilon\|^{p-1}_{0,\alpha,\nu}\\ &+(1+\eta)pC_\eta\ \|v_1\|^{p-1}_{0,\alpha,\nu}+C_\eta\ \|v_1-v_2\|^{p-1}_{0,\alpha,\nu}]\\ &+\|v_1-v_2\|_{0,\alpha,\nu}\,[(1+\eta)^2p\ \|\tilde{u}_\varepsilon\|^{p-1}_{0,\alpha,\nu}\\ &+(1+\eta)pC_\eta\ \|v_1\|^{p-1}_{0,\alpha,\nu}+C_\eta\ \|v_1-v_2\|^{p-1}_{0,\alpha,\nu}]),\end{aligned}$$

which implies that

$$\begin{aligned}\|\mathbb{P}(v_1)-\mathbb{P}(v_2)\|_{2,\alpha,\nu} &\leqslant C\,\|v_1-v_2\|_{0,\alpha,\nu}\,(\|\|\mathbb{L}\|\|+1)\\ &\quad\times[(1+\eta)^2\,pc_p+(1+\eta)\,pC_\eta\theta^{p-1}+C_\eta(2\theta)^{p-1}].\end{aligned}$$

For the values of p very close to $m/(m-2)$ the quantity in the right hand side is less than 1 so that the operator becomes a contraction. So $\mathbb{P}$ has a fixed point u in B_θ thanks to the Schauder fixed point theorem (cf. [8]). We conclude that there exists u satisfying $\Delta(u+\tilde{u}_\varepsilon)+|u+\tilde{u}_\varepsilon|^p=0$ and such that $u+\tilde{u}_\varepsilon$ has the same singularities as $\tilde{u}_\varepsilon$, namely those prescribed by Σ. As we have $|u|\leqslant r^\nu$ and $u_0\equiv r^{-2/(p-1)}$ in the neighborhood of Σ, we deduce that $\tilde{u}_\varepsilon+u>0$ in the neighborhood of Σ. On the other hand, knowing that $-\Delta(u+\tilde{u}_\varepsilon)\geqslant 0$ in Ω, the maximum principle (cf. [8]) implies that $(u+\tilde{u}_\varepsilon)\geqslant 0$ in Ω and using the strong maximum principle of Hopf (cf. [8], [1]) and the condition of preservation of the singularities of $\tilde{u}_\varepsilon$ we can confirm that we have $(u+\tilde{u}_\varepsilon)>0$ in Ω and that it is a solution to the problem (P_Σ). Letting ε tend to zero, we obtain a function $\tilde{u}_0$ satisfying $-\Delta(u+\tilde{u}_0)=(u+\tilde{u}_0)^p$ in Ω and $u+\tilde{u}_0=0$ on $\partial\Omega$. This ends the proof of the Theorem 1.

Acknowledgments

I am grateful to F. Pacard for his helpful advice and encouragement. I also thank the referee for helpful remarks.

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